

Batalin–Vilkovisky Formalism and Integration Theory on Manifolds

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The correspondence between the BV-formalism and integration theory on supermanifolds is established. An explicit formula for the density on a Lagrangian surface in a superspace provided with an odd symplectic structure and a volume form is proposed.

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1 Introduction

In their outstanding works [1] Batalin and Vilkovisky proposed the most general method for quantizing arbitrary gauge field theories.

During the years it becomes clear that this scheme is very powerful for resolving ghost problems and moreover it contains a rich geometrical structure. In the paper [2] Witten proposed a program for the construction of String Field Theory in the framework of the Batalin-Vilkovisky formalism (BV-formalism) and noted the necessity of its geometrical investigation. The BV-formalism indeed uses the geometry of the superspace provided with odd symplectic structure and the volume form. The properties of this geometry and its connection to the BV formalism was investigated for example, in [3,4,5,6]. Particularly in [5] A. S. Schwarz gives the detailed geometrical analysis of the BV-formalism in terms of this geometry.

However, some specific aspects of the BV-formalism are not completely clarified, such as:

- the geometrical meaning of the initial conditions of the master-action,
- the choice of the gauge fermion and the geometrical reasons for the extending the initial space of fields with ghosts and antighost fields.

In this work we try to analyze some of these questions. For this purpose we study the analogy between the BV-scheme and the corresponding constructions in differential geometry.

From the geometrical point of view to the gauge symmetries correspond the vector fields on the space of the classical fields which preserve the action. The partition function, when gauge conditions are fixed, is the integral of a nonlocal density constructed by means of these vector fields over the surface which is defined by gauge conditions. This surface is embedded in the space of the classical fields.

The gauge independence means that this density have to be closed. To make this density local in the BV formalism one have to rise the density and the gauge fixing surface on the extended space: to the gauge fixing surface corresponds the Lagrangian manifold embedded in the phase space of the "fields" and "antifields" ("fields" = classical fields, ghosts), to the closed density corresponds the volume form on this manifold (the exponent of the BV master-action) which obeys to BV master equation [1,5,6].

In the 2-nd Section we briefly recall the basic formulae of the BV formalism and following [5] give the covariant explicit formula for the volume element on the Lagrangian manifold when it is given by arbitrary functions of the fields and antifields. This formula is related to the multilevel field-antifield formalism with the most general Lagrangian hypergauges [11].

In the 3-th Section we briefly recall the basic constructions of the geometry of the superspace provided with an odd symplectic structure and volume form [3,5,6].— It is this geometry on which the BV formalism is based, and which development on the other hand was highly inspired by this formalism. In particular we shortly describe the properties of the Δ -operator arising in this geometry and the connection between the BV-formalism and the Δ -operator nilpotency condition.

In the 4-th Section we consider the densities [7,8,9,10] (the general covariant objects which can be integrated over supersurfaces in the superspace). Following [8] and [10] we consider a special class of densities — pseudodifferential forms on which the exterior derivative can be defined correctly. Using Baranov-Schwarz (BS) transformations [8] we rise these forms to integration objects on the enlarged space and formulate the condition of closure of these forms in terms of the Δ -operator.

In the 5-th Section, using BS transformations we study the relations between gauge symmetries in field theory and the closed pseudodifferential forms corresponding to the integrand for the partition function of the theory. We study the relations between the closure conditions and the BV-master-equation.

2. BV Formalism

In this section we recall the basic constructions of BV formalism [1]: the integral for the partition function and we rewrite this integral in the case where the lagrangian manifold is given in covariant way.

Let $S(\phi)$ be the action of theory with gauge symmetries $\{R_b^A(\phi)\}$:

$$R_b^A(\phi) \frac{\delta S(\phi)}{\delta \phi^A} = 0 \quad . \quad (2.1)$$

We use de Witt condensed notations (index A runs over the all the indices and the spatial coordinates of the fields ϕ). Let \mathcal{E} be the space of the fields Φ^A and antifields Φ_{*A} where $\Phi^A = (\phi^A, c^b, \nu_b, \dots)$ is the space of fields ϕ^A enlarged with the ghosts, lagrangian multipliers for the constraints e.t.c. and Φ^{*A} has the parity opposite to Φ^A

$$p(\Phi^{*A}) = p(\Phi^A) + 1 \quad (2.2)$$

In the space \mathcal{E} one can define the symplectic structure by the odd Poisson bracket:

$$\{F, G\} = \frac{\delta F}{\delta \Phi^A} \frac{\delta G}{\delta \Phi_{*A}} + \frac{\delta F}{\delta \Phi_{*A}} \frac{\delta G}{\delta \Phi^A} \quad (\text{if } F \text{ is even}) \quad (2.3)$$

and the Δ_0 operator:

$$\Delta_0 F = \frac{\delta^2 F}{\delta \Phi_{*A} \delta \Phi^A} \quad (2.4)$$

The master action \mathcal{S} then can be uniquely defined by the equation

$$\Delta_0 e^{\mathcal{S}} = 0 \Leftrightarrow \Delta_0 \mathcal{S}(\Phi^A, \Phi_{*A}) + \frac{1}{2} \{ \mathcal{S}(\Phi^A, \Phi_{*A}), \mathcal{S}(\Phi^A, \Phi_{*A}) \} = 0 \quad (2.5)$$

and the initial conditions:

$$\mathcal{S}(\Phi^A, \Phi_{*A}) = S(\phi) + c^b R_b^A \phi_{*A} + \dots \quad (2.5a)$$

Where dots means terms containing ghosts and antifields of higher degrees.

If

$$[\mathbf{R}_a, \mathbf{R}_b] = t_{ab}^c \mathbf{R}_c + E_{ab}^{[AB]} \mathcal{F}_B$$

where \mathcal{F}_A are the equations of motion ($\mathcal{F}_A = \frac{\delta S(\phi)}{\delta \phi^A}$), then,

$$\mathcal{S}(\Phi^A, \Phi_{*A}) = S(\phi) + c^b R_b^A \phi_{*A} + \frac{1}{2} t_{ab}^c c^a c^b c_c^* + \frac{1}{2} c^a c^b E_{ab}^{CD} \phi_{*C} \phi_{*D} + \dots \quad (2.5b)$$

To the gauge conditions

$$f_b = 0 \quad (2.6)$$

corresponds the so called "gauge fermion":

$$\Psi = f_b \nu^b \quad (2.7)$$

which defines the Lagrangian surface Λ in \mathcal{E} by the equations

$$F_A(\Phi^A, \Phi_{*A}) = 0, \quad (2.8)$$

where

$$F_A = \Phi_{*A} - \frac{\delta \Psi(\Phi)}{\delta \Phi^A} = 0 \quad (2.9)$$

(the surface embedding in the symplectic space is Lagrangian if it has half the dimension of space and the two-form defining the symplectic structure is equal to zero on it). The partition function Z is given by the integral of the master-action exponent over this Lagrangian surface Λ :

$$Z = \int e^{\mathcal{S}(\Phi_A, \Phi^{*A})} \delta(\Phi_{*A} - \frac{\delta \Psi(\Phi)}{\delta \Phi^A}) \mathcal{D}\Phi^* \mathcal{D}\Phi \quad (2.10)$$

(See for details [1]).

The main statement of the BV formalism is that this integral does not depend on the choice of the Lagrangian surface Λ .

Before going into the geometrical analysis of the formula (2.10) we first rewrite it in a more covariant way if the functions F_a which define Λ by the equation (2.8) are arbitrary.

It is easy to see that the surface Λ defined by (2.8) is Lagrangian iff

$$\left. \{F_A, F_B\} \right|_{F_A=0} = 0 \quad (2.11)$$

Let us consider the integral:

$$\int e^{\mathcal{S}(\Phi_A, \Phi^{*A})} \sqrt{\text{Ber} \frac{\delta(G^A, F_B)}{\delta(\Phi^A, \Phi_{*A})}} \sqrt{\text{Ber}\{G^{\tilde{A}}, F_B\} \delta(F)} \mathcal{D}\Phi^* \mathcal{D}\Phi \quad (2.12)$$

where G^A are arbitrary functions and \tilde{A} has a parity reversed to A .

One can show that if the functions F_A define the Lagrangian manifold Λ (2.8) then this integral does not depend on the choice of the functions G_A and it does not depend on the choice of the functions F_A defining Λ . On the other hand in the case where the functions F_A have the form (2.9) and the functions G^A are equal to Φ^A , it evidently coincides with the BV integral (2.10).

3 The survey of BV formalism geometry

The formulae (2.5—2.12) of the previous section have the following geometrical meaning (see for details [3,5,6] and also [12]). In the superspace $E^{(n,n)}$ with the coordinates

$z^A = (x^1, \dots, x^n, \theta^1, \dots, \theta^n)$ where x^i are even, θ^i odd coordinates one can consider the structure defined by the pair $(\rho, \{, \})$, where ρ is the volume form and $\{, \}$ the odd nondegenerated Poisson bracket corresponding to the odd symplectic structure. To the structure $(\rho, \{, \})$ on $E^{(n,n)}$ corresponds the following geometrical constructions which consistue the essence of BV formalism geometry.

We define a second order differential operator on E (so called Δ -operator)

$$\Delta_\rho f = \frac{1}{2} \text{div}_\rho \mathbf{D}_f \equiv \frac{1}{2} \frac{\mathcal{L}_{\mathbf{D}_f} \rho}{\rho}, \quad (3.1)$$

where \mathbf{D}_f is the Hamiltonian vector field corresponding to the function f . This operator is typical for the odd symplectic geometry[3]. If $\rho = \rho(z) d^n x d^n \theta$ then

$$\Delta_\rho f = \frac{1}{2\rho} (-1)^{p(A)} \frac{\partial}{\partial z^A} (\rho \{z^A, f\}) = \frac{1}{2} \{ \log \rho, f \} + \frac{\partial^2 f}{\partial x^i \partial \theta^i}, \quad (3.2)$$

where $p(A)$ is the parity of the coordinate z^A .

We say that the pair $(\rho, \{, \})$ is canonical in the coordinates

$z^A = (x^1, \dots, x^n, \theta^1, \dots, \theta^n)$ if $\rho = 1 \cdot d^n x d^n \theta$ and if the Poisson bracket is canonical one:

$$\{f, g\} = \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial \theta^i} + (-1)^{p(f)} \frac{\partial f}{\partial \theta^i} \frac{\partial g}{\partial x^i}. \quad (3.3)$$

Then the Δ -operator takes the canonical expression:

$$\Delta_0 f = \frac{\partial^2 f}{\partial x^i \partial \theta^i}. \quad (3.4)$$

If two Δ -operators Δ_ρ and $\Delta_{\tilde{\rho}}$ correspond to two structures with the different volume forms ρ and $\tilde{\rho}$ and the same symplectic structure¹. then it is easy to see using (3.2) that

$$\Delta_{\tilde{\rho}} f = \Delta_\rho f + \frac{1}{2} \{ \log \lambda, f \}, \quad (3.5)$$

¹ Indeed because of Darboux theorem we can always consider (at least locally) the canonical symplectic structure (3.3)

and

$$\Delta_{\tilde{\rho}}^2 f = \Delta_{\rho}^2 f + \{\lambda^{-\frac{1}{2}} \Delta_{\rho} \lambda^{\frac{1}{2}}, f\}. \quad (3.6)$$

where $\tilde{\rho} = \lambda \rho$.

For a given structure $(\rho, \{, \})$ the following statements are equivalent:

i) the operator Δ_{ρ} is nilpotent

$$\Delta_{\rho}^2 = 0, \quad (3.7i)$$

ii) the function $\rho(z)$ defining the volume form ρ obeys the equation:

$$\Delta_0 \sqrt{\rho} = 0 \quad (3.7ii)$$

iii) there exist coordinates in which the pair $(\rho, \{, \})$ is canonical.²

The iii) \Rightarrow i) is evident, the i) \Leftrightarrow ii) immediately follows from (3.6). The i) \Rightarrow iii) needs more detailed analysis.

The pair $(\rho, \{, \})$ generates the invariant volume form ρ_{Λ} on arbitrary Lagrangian manifolds Λ in E —”the square root of the volume form ρ ” in the following way [5]:

$$\rho_{\Lambda}(e_1, \dots, e_n) = \sqrt{\rho(e_1, \dots, e_n, f_1 \dots, f_n)} \quad (3.8)$$

where $\{e_i\}$ are the vectors tangent to Λ and $\{f_i\}$ are arbitrary vectors such that

$$w(e_i, f_j) = \delta_{ij}.$$

In these terms the BV formalism has the following geometrical meaning: We consider in the superspace \mathcal{E} of the fields and antifields the pair $(\rho, \{, \})$ where the volume form is defined by the master-action:

$$\rho = e^{2\mathcal{S}}, \quad (3.9)$$

and $\{, \}$ is defined by (2.3). Then using i), ii), iii) and comparing formulae (3.7) with formulae (2.3–2.5) we see that the master-equation is nothing but the condition of nilpotency of the corresponding Δ operator. The partition function is nothing but the integral of the invariant volume form (3.8) on the Lagrangian surface Λ [5] and the eq. (2.12) is the covariant expression for this volume form.

In the next section we will try to understand these statements from the point of view of integration theory on surfaces.

² The structures $(\rho, \{, \})$ for which these properties are obeyed are called SP structures [5]. One of us (O.M.K.) wants to note that in [3] where was first introduced the Δ -operator related to the structure $(\rho, \{, \})$ for an arbitrary volume form in superspace the false statement that every $(\rho, \{, \})$ structure is SP structure was made

4 Integration over surfaces

In this section we present the basic objects of integration theory on supermanifolds: densities and dual densities [8–10]. We consider the special class of densities on which the exterior derivative can be defined correctly—pseudodifferential forms [7–10]. Then we describe the Baranov–Schwarz (BS) representation of the pseudodifferential forms via the function on the superspace associated to the tangent bundle of initial space [8]. Considering the dual construction we show that the closure of the pseudodifferential form in the BS representation is formulated in terms of the Δ operator.

Densities

Let Ω be an arbitrary supersurface in the superspace E with coordinates z^a , given by a parametrization $z^a = z^a(\zeta^s)$. The function $L(z^a, \frac{\partial z^a}{\partial \zeta^s})$ on E is called a density (covariant density), if it satisfies the condition [9]:

$$L(z^a, \frac{\partial z^a}{\partial \zeta^{s'}} K_s^{s'}) = L(z^a, \frac{\partial z^a}{\partial \zeta^s}) \text{Ber} K_s^{s'}, \quad (4.1)$$

where Ber is the superdeterminant of the matrix.

Then the following integral does not depend on the choice of the parametrization of the surface Ω

$$\Phi_\Omega(L) = \int L(z^a(\zeta), \frac{\partial z^a(\zeta)}{\partial \zeta^s}) d\zeta, \quad (4.2)$$

and correctly defines the functional on the surface Ω corresponding to the density L .

In the bosonic case where there are not odd variables, one can see that if a density L is a linear function of the $\frac{\partial z^a(\zeta)}{\partial \zeta^s}$ then to L corresponds a differential form. The covariant density is closed if it satisfies identically the condition:

$$\Phi_{\Omega+\delta\Omega}(L) = \Phi_\Omega(L) \quad (4.3)$$

for an arbitrary variation of an arbitrary surface Ω (up to boundary terms).

It is easy to see that

$$\Phi_{\Omega+\delta\Omega}(L) - \Phi_\Omega(L) = \mathcal{F}_a(z) \delta z^a \quad (4.4)$$

where

$$\mathcal{F}_a(z) = \frac{\partial L}{\partial z^a} - (-1)^{p(a)p(s)} \frac{d}{d\zeta^s} \frac{\partial L}{\partial z_{,s}^a} \quad (4.5)$$

are the left part of the Euler-Lagrange equations of the functional $\Phi(L)$.

How to define exterior derivative operator on the densities?

If d is the exterior derivative, then

$$\Phi_{\Omega+\delta\Omega}(L) - \Phi_\Omega(L) = \Phi_{\delta V}(dL) \quad (\text{Stokes theorem}). \quad (4.6)$$

Eq.(4.6) put strong restrictions on the class of densities on which the operator d is correctly defined [10]. Comparing (4.4), (4.5) and (4.6) we see that d is correctly defined if $\mathcal{F}_a(z)$ in (4.5) *do not contain the second derivatives of ζ* [10]:

$$\frac{\partial^2 L}{\partial z_{,s}^a \partial z_{,t}^b} = -(-1)^{p(s)p(t)+(p(s)+p(t))p(b)} \frac{\partial^2 L}{\partial z_{,t}^a \partial z_{,s}^b}. \quad (4.7)$$

In this case dL defined by (4.6) does not depend on the second derivatives and

$$d^2 = 0 \quad . \quad (4.8)$$

The densities, which obey the conditions (4.7) are called pseudodifferential forms.

In the bosonic case from (4.7) follows that the density is a linear function of the variables $\frac{\partial z^a(\zeta)}{\partial \zeta^s}$ i.e. the exterior derivation can be defined only on the densities which correspond to the differential forms. In the supercase in general from (4.7) linearity conditions do not follow—the differential forms in the superspace are not in general integration objects over supersurfaces. It is the pseudodifferential forms which take their place as integration objects obeying Stokes theorem [7–10]).

To obtain the pseudodifferential forms, Baranov and Schwarz in [8] suggested the following procedure which seems very natural in the spirit of a ghost technique:

Let STE be the superspace associated to the tangent bundle TE of the superspace E and (z^a, z^{*a}) its (local) coordinates. The coordinates z^{*a} transform from map to map like dz^a , and their parity is reversed: $p(z^{*a}) = p(z^a) + 1$. Then to an arbitrary function $W(z^a, z^{*a})$ on STE corresponds the density:

$$L_W = L(z^a, \frac{\partial z^a}{\partial \zeta^s}) = \int W(z^a, \frac{\partial z^a}{\partial \zeta^s} \nu^s) d\nu \quad , \quad (4.9)$$

where ν^s has the reversed parity:

$$p(\nu^s) = p(\zeta^s) + 1. \quad (4.10)$$

It is easy to see using (4.10) that (4.9) obeys equations (4.1) and (4.7) so that equation (4.9) indeed defines a density which is a pseudodifferential form. We say that the function W is the BS representation of the pseudodifferential form L_W .

A simple calculations show that in the BS representation the exterior differentiation operator has the following expression:

$$\hat{d} = (-1)^{p(a)} z^{*a} \frac{\partial}{\partial z^a} \quad (4.11)$$

$$(d(L_W) = L_{\hat{d}W}).$$

Dual densities

Consider now the dual constructions.

Let E be the superspace, and $\rho(z)dz$ the volume form is defined on it.

Let Ω be an arbitrary supersurface in the superspace E with coordinates z^a , given not by the parametrization $z^a = z^a(\zeta^s)$ but by the equations

$$f^\alpha(z) = 0. \quad (4.12)$$

The function $\tilde{L} = \tilde{L}(z^a, \frac{\partial f^\alpha}{\partial z^a})$ is called a D-density (dual density) if it is satisfied to the condition :

$$\tilde{L}(z^a, \frac{\partial f^\alpha(z)}{\partial z^a} \eta_\alpha^\beta) = \tilde{L}(z^a, \frac{\partial f^\alpha(z)}{\partial z^a}) \text{Ber } \eta_\alpha^\beta. \quad (4.13)$$

Then the following integral does not depend on the choice of the equations (4.12) which define the surface Ω

$$\Phi_\Omega(\tilde{L}) = \int \tilde{L}(z^a, \frac{\partial f^\alpha(z)}{\partial z^a}) \delta(f^\alpha(z)) \rho(z) dz, \quad (4.14)$$

and correctly defines the functional on the surface Ω corresponding to the D-density \tilde{L} .

The D-density \tilde{L} corresponds to the density L ($\tilde{L} \rightarrow L$) if for the arbitrary surface Ω the functionals (4.2) and (4.14) coincide. (See for the details [9])

(For example the integrand in (2.12) is a D-density which corresponds to the density (3.8))

The D-density is closed, if it satisfies the condition (4.3) (where we replace $\tilde{L} \rightarrow L$).

One can obtain the dual densities corresponding to pseudodifferential forms (such densities are called pseudointegral forms) by the procedure dual to the Baranov-Schwarz one:

Let ST^*E be the superspace associated to the cotangent bundle T^*E of the superspace E and (z^a, z_a^*) its (local) coordinates. The coordinates z_a^* transform from map to map like $\frac{\partial}{\partial z^a}$, and their parity is reversed: $p(z_a^*) = p(z^a) + 1$. Then to an arbitrary function $W(z^a, z_a^*)$ on ST^*E corresponds the D-density—pseudointegral form:

$$\tilde{L}_W = \tilde{L}(z^a, \frac{\partial f^\alpha}{\partial z^a}) = \int W(z^a, \frac{\partial f^\alpha}{\partial z^a} \nu_\alpha) d\nu. \quad (4.15)$$

where ν^α have the reversed parity like in (4.10):

$$p(\nu^\alpha) = p(f^\alpha) + 1.$$

The functional (4.14) can be expressed in terms of the function W in the following way:

$$\Phi_\Omega(\tilde{L}) = \int \rho(z) W(z, z^*) \delta(z_a^* - \frac{\partial f^\alpha}{\partial z^a} \nu_\alpha) \delta(f^a) dz dz^* d\nu. \quad (4.15a)$$

A straightforward calculation show that the operator of exterior differentiation \hat{d} in the BS representation of the pseudointegral forms has the following expression:

$$\hat{d} = \frac{1}{\rho} \frac{\partial \rho}{\partial z^a} \frac{\partial}{\partial z_a^*} + \frac{\partial^2}{\partial z^a \partial z_a^*}. \quad (4.16)$$

$$(\text{If } \tilde{L} = \tilde{L}_W \rightarrow L \text{ then } \tilde{L}' = \tilde{L}_{\hat{d}W} \rightarrow dL).$$

Comparing the equations (4.16) and (3.2), we see that on the superspace ST^*E it is natural to consider the structure $(\hat{\rho}, \{, \})$ (See the Sect.3) where $\{, \}$ is the canonical odd symplectic structure on ST^*E generated by the relations

$$\{z^a, z_b\} = \{z^{*a}, z_b^*\} = 0, \quad \{z^a, z_b^*\} = (-1)^{p(a)} \delta_b^a \quad (4.17)$$

and the volume form

$$\hat{\rho} = \rho^2 (z^1 \cdots z^n) dz^1 \dots dz^n dz_1^* \dots dz_n^*. \quad (4.18)$$

(One can note that (4.18) is in the accordance with (3.8).—The space E with volume form ρ is evidently the Lagrangian surface in ST^*E with volume form (4.18))

Comparing (4.16) and (3.2) we see that to the operator of the exterior differentiation corresponds the Δ –operator:

$$\hat{d} = \Delta_{\hat{\rho}} \quad (4.19)$$

and the condition of closure of the dual density \tilde{L}_W in the BS representation is

$$\Delta_{\hat{\rho}} W = 0, \quad (4.19a)$$

where $\hat{\rho}$ is defined by (4.18) and $\Delta_{\hat{\rho}}$ by (3.2). This operator in this case is nilpotent because it corresponds to exterior differentiation operator. (Independently from (4.16) and (4.8) it follows from (4.18) and (3.7ii) or from (4.18) and (3.7iii) because $\hat{\rho}$ depends on the half of the variables of the superspace ST^*E .)

5 The closed densities and the BV formalism geometry.

In this section we consider two examples of the previous constructions comparing them with the constructions of the Sections 2,3 and 4. We check connections between the gauge symmetries of the theory, the densities which are integrand in the partition function after eliminating gauge degrees of freedom, and volume forms obeying the BV–master–equation.

Example 1 Let $R^a(z^a) \frac{\partial}{\partial z^a}$ be an even vector field on the superspace E with coordinates (z^1, \dots, z^n) and with volume form $\rho = \rho(z) dz^1 \cdots dz^n$. To this vector field corresponds the D–density

$$\tilde{L} = R^a(z^a) \frac{\partial f}{\partial z^a}. \quad (5.1)$$

One can define the functional on the surfaces of codimension (1.0) corresponding to the density (5.1):

$$\Phi_{\Omega}(\tilde{L}) = \int \tilde{L}(z^a, \frac{\partial f(z)}{\partial z^a}) \delta(f(z)) \rho(z) dz = \int R^a(z) \frac{\partial f(z)}{\partial z^a} \delta(f(z)) \rho(z) dz, \quad (5.2)$$

where $f = 0$ is the equation which defines the surface Ω (f is an even function). This functional is nothing but the well-known formula for the flux of the vector field through the surface Ω . It is evident that the density \tilde{L} in (5.1) is pseudointegral form. To this density corresponds the function (4.15)

$$W = (-1)^{p(a)} R^a(z) z_a^* \quad (5.3)$$

on ST^*E . ($\tilde{L} = \tilde{L}_W$). The condition of closure of the density (5.1) is the Gauss formula:

$$\text{div}_{\rho} \mathbf{R} = \frac{1}{\rho} (-1)^a \frac{\partial(\rho R^a)}{\partial z^a} = 0. \quad (5.4)$$

In BS representation it is (4.18, 4.19)

$$\Delta_{\hat{\rho}} W = \Delta_{\rho^2} W = 0. \quad (5.5)$$

We can consider this example as a toy example of field theory.

Let a space E be the space of fields configurations ($z^a \rightarrow \varphi^a$) Let $R^a(z) \frac{\partial}{\partial z^a}$ be the "gauge" symmetry of the action $S(z)$ (compare with (2.1)):

$$R^a(z) \frac{\partial S(z)}{\partial z^a} = 0 \quad (5.6)$$

and this symmetry preserves the canonical volume form:

$$(-1)^{p(a)} \frac{\partial R^a}{\partial z^a} = 0. \quad (5.7)$$

If we put

$$\rho = e^S \quad (5.8)$$

then we see that the functional (5.2) corresponding to the density (5.1) constructed via the "gauge symmetry" \mathbf{R} is the partition function of the theory with the action S after eliminating the "gauge" degrees of freedom corresponding to the symmetry \mathbf{R} . From (5.6–5.8)) follow (5.4, 5.5) hence (5.1) is closed and (5.2) is "gauge" independent.

Now we consider the more realistic

Example 2 Let

$$\{\mathbf{R}_{\alpha} = R_{\alpha}^a(z) \frac{\partial}{\partial z^a}\}, \quad (\alpha = 1, \dots, m) \quad (5.9)$$

be the collection of the vector fields on the superspace E with coordinates (z^1, \dots, z^n) and with volume form

$$\rho = \rho(z) dz^1 \cdots dz^n. \quad (5.10)$$

To (5.9) corresponds D-density

$$\tilde{L} = Ber(R_\alpha^a(z) \frac{\partial f^\beta}{\partial z^a}) \quad (5.11)$$

(the condition (4.13) is evidently satisfied.) One can consider the functional:

$$\Phi_\Omega(\tilde{L}) = \int Ber(R_\alpha^a(z) \frac{\partial f^\beta}{\partial z^a}) \rho(z) \delta(f) d^n z \quad (5.12)$$

where Ω is the surface defined by the equations

$$f^\alpha = 0.$$

(In the usual (not super)case, (5.12) can be considered as the flux of the polivectorial field $\mathbf{R}_1 \wedge \cdots \wedge \mathbf{R}_m$ through the surface Ω .)

One can see that \tilde{L} in (5.11) is pseudointegral form \tilde{L}_W where the W —BS representation of this density can be defined by the following formal relation:

$$W = \int e^{c^\alpha R_\alpha^a(z) z_a^*} d\mathbf{c} \quad (5.13)$$

where we introduce additional variables (ghosts) c^α ($p(c^\alpha) = p(\nu^\alpha)$). ((5.13) is correct if all the symmetries \mathbf{R}_α are even).

Let the equations (5.6), (5.7) be satisfied for all \mathbf{R}_α — these vector fields being the gauge symmetries of the theory with the action S . Again as in the Example 1 we consider as volume form the exponent of the action (5.8). Does the density (5.11) is closed in this case?

It is easy to see that

$$\forall R^a(z) \frac{\partial S(z)}{\partial z^a} = 0 \rightarrow [\mathbf{R}_\alpha, \mathbf{R}_\beta] = t_{\alpha\beta}^\gamma \mathbf{R}_\gamma + E_{\alpha\beta}^{[ab]} \frac{\partial S(z)}{\partial z^b}. \quad (5.14)$$

To check the relation with the BV-formalism we consider instead superspace E the superspace E^e enlarged with the additional coordinates c^α . (The coordinates of E^e are $z^A = (z^a, c^\alpha)$). The volume forms $\rho(z)$ on E and $\hat{\rho}$ on ST^*E (see (4.18)) and the symplectic structure (4.17) are naturally prolonged on E^e and ST^*E^e .

Using (4.15, 4.15a) and (5.13) we rewrite (5.12) as the integral over the space T^*SE^e :

$$\Phi_\Omega(\tilde{L}) = \int \rho e^{c^\alpha R_\alpha^a(z) z_a^*} d\mathbf{c} \delta(z_a^* - \frac{\partial f^\alpha}{\partial z^a} \nu_\alpha) \delta(f) dz dz^* d\nu = \quad (5.15)$$

$$\int \rho W^e(z^A, z_A^*) \delta(z_A^* - \frac{\partial f^\alpha}{\partial z^A} \nu_\alpha) dz dz^* d\nu, \quad (5.16)$$

where

$$W^e(z^A, z_A^*) = e^{c^\alpha R_\alpha^a(z) z_a^*} \quad (5.17)$$

is the BS representation of the pseudointegral form in ST^*E^e . Using (4.19) we can check its closure.

((5.15, 5.16) is the partition function of the theory obtained after performing the Fadeev–Popov trick).

Let

$$\Delta_{\hat{\rho}} W^e = \Delta_{\hat{\rho}} e^{c^\alpha R_\alpha^a(z) z_a^*} = 0 \quad (5.18)$$

be satisfied. The condition (5.18) means that not only the function W on ST^*E corresponds to the closed density on E (i.e.the partition function (5.15) is gauge invariant) but the function W^e on ST^*E^e corresponds to the closed density on E^e as well. In this case from (3.6) and (3.7) follows that the Δ operator corresponding to the volume form

$$\hat{\rho}' = \hat{\rho} \cdot (W^e)^2 \quad (5.19)$$

is nilpotent (W^e in contrary to W is even) as well as the Δ operator corresponding to the volume form (5.8). Now from (3.7) follows that the master-action \mathcal{S}

related with ρ' in the same way as S is related with ρ in (5.8):

$$\mathcal{S} = S + c^\alpha R_\alpha^a z_a^*, \quad (\rho' = e^{\mathcal{S}}) \quad (5.20)$$

obeys the master-equation. So in the case where (5.18) holds, starting from gauge symmetries we constructed the closed density (5.12, 5.13), interpreting the volume form as the exponent of the action. The corresponding functional (5.12) is the partition function. Localizing this density in the space enlarged with the ghosts we came to the volume form (exponent of the master action) which obeys to the master-action.

In general case the density (5.11) *is not closed* and the partition function (5.12, 5.16) is not gauge invariant.

Even in the case where the algebra of the symmetries is closed:

$$t_{\alpha\beta}^\gamma = \text{const} \text{ and } E_{\alpha\beta}^{[ab]} \equiv 0 \quad (5.21)$$

the application of the Δ -operator (4.19) to (5.17) and (5.13) give us

$$\Delta_{\rho^2} W^e = \Delta_{\rho^2} e^{c^\alpha R_\alpha^a(z) z_a^*} dc = \frac{1}{2} c^\alpha c^\beta (t_{\alpha\beta}^\gamma R_\gamma^a(z) + E_{\alpha\beta}^{[ab]} \frac{\partial S(z)}{\partial z^b}) z_a^* e^{c^\alpha R_\alpha^a(z) z_a^*} dc \quad (5.22)$$

and

$$\Delta_{\rho^2} W = \Delta_{\rho^2} \int e^{c^\alpha R_\alpha^a(z) z_a^*} dc = \int c^\alpha c^\beta (t_{\alpha\beta}^\gamma R_\gamma^a(z) + E_{\alpha\beta}^{[ab]} \frac{\partial S(z)}{\partial z^b}) z_a^* e^{c^\alpha R_\alpha^a(z) z_a^*} dc. \quad (5.23)$$

In particular it is easy to see from (5.22) that if the algebra of the symmetries is abelian we come to (5.18).

If, for example, the symmetries are even and they consist the closed unimodular algebra ($E_{\alpha\beta}^{[ab]} = 0$, $t_{\alpha\beta}^\gamma = \text{const}$ and $\sum_\alpha t_{\alpha,\beta}^\alpha = 0$) then the right hand side of (5.23) is vanishing, so the function W corresponds to closed density in E (i.e. the partition function is gauge invariant). But the function W^e in (5.22) does not correspond to closed density in E^e . To close it in this case one have to consider in the space ST^*E^e the function

$$W^{e'}(z^A, z_A^*) = e^{c^\alpha R_\alpha^a(z) z_a^* + \frac{1}{2} t_{\alpha\beta}^\gamma c^\alpha c^\beta c_\gamma^*}$$

which corresponds to a closed density in E^e . So the corresponding volume form and the master action

$$\mathcal{S} = S + c^\alpha R_\alpha^a z_a^* + \frac{1}{2} t_{\alpha\beta}^\gamma c^\alpha c^\beta c_\gamma^*$$

obey the master equation (compare with (2.5b)).

In the general case the density (5.11, 5.13) plays the role of initial conditions for constructing the closed density in enlarged space—i.e. the volume form (the exponent of the master-action) obeying the (3.7).

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References

- [1] Batalin I.A., Vilkovisky G.A. –Phys. Lett., **102B**, 27 (1981);
Phys.Rev **D28** 2567 (1983); Nucl.Phys. **B234** 106 (1984)
- [2] E. Witten—Mod. Phys. Lett. **A5** 487 (1990)
- [3] Khudaverdian O.M—J. Math. Phys., **32** 1934 (1991):
Preprint of Geneva Universite UGVA-DPT 1989/05–613 .
- [4] M. Henneaux –Phys. Lett., **B315**, 283 (1993)
H.Hata, B. Zwiebach—Annals of Phys., **229**, 177 (1994)
J.Alfaro, P.H.Damgaard—Nucl.Phys., **B404**, 751 (1993)
- [5] Schwarz A.S. - Comm. Math. Phys. **155**,249 (1993)
- [6] O. M. Khudaverdian , A. P. Nersessian— Mod. Phys. Lett **A8**, 2377 (1993);
J.Math.Phys., **34** 5533 (1993)
- [7] J.N. Bernstein, D.A.Leitec — Funk.Anal.i ego pril. **11**,No1, 55 (1977);
ibid **11**, No3, 70 (1977).
- [8] M. Baranov, A. S. Schwarz – Funk. anal i ego pril. **18**, No.2 ,53 (1984);
ibid **18**, No.3, 69 (1984)
- [9] A.Gayduk, O.M. Khudaverdian, A.S.Schwarz— Teor. i Mat. Fizika **52**, 375 (1982)
- [10] T. Voronov - Geometric integration theory over supermanifolds.—
Sov. Scient. Rev.C Math.Phys., Vol.9, pp.1–138 (1992)
- [11] I.A. Batalin, I.V. Tyutin— Mod.Phys.Lett., **A8**, 283 (1993); *ibid* **A8**, 3673 (1993);
ibid **A9**, 1707 (1994) (Preprint FIAN/TD/4–94)
- [12] I.A. Batalin, I.V. Tyutin—Int. J. Mod.Phys. **A8** 2333 (1993)